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A homogenization theory for elastic–viscoplastic composites with point symmetry of internal distributions

N. Ohno [a,*], T. Matsuda [a], X. Wu [b]

[a] Department of Mechanical Engineering, School of Engineering, Nagoya University, Chikusa-ku, Nagoya 464-8603, Japan

[b] Rock Engineering Department, OYO Corporation, 2-61-5 Toro-cho, Omiya, Saitama 330-8632, Japan

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Abstract

The homogenization theory for nonlinear time-dependent composites with periodic internal structures developed by the present authors is rebuilt on the assumption that constituents, stress and strain distribute point symmetrically with respect to the center of each unit cell in periodic composites. It is proved that this symmetry also allows the field of perturbed velocity to satisfy the symmetry with respect to the cell boundary facet centers. The symmetry is then used for imposing the boundary condition on perturbed velocity and for developing the theory in an integral form in which a half of the unit cell is taken as the domain of analysis. As applications of the theory, elastic–viscoplastic elongation of unidirectional composites is analyzed by assuming the square and hexagonal arrays of fibers subjected to either transverse or off-axial loading. It is thus shown that the point symmetry can be used for computing efficiently the inelastic behavior of nonlinear, periodic composites.

Keywords: Homogenization; Boundary conditions; Symmetric; Viscoplastic

1. Introduction

The homogenization method based on the two-scale asymptotic expansion of field variables was developed for composites with periodic and quasi-periodic internal structures (Babuska, 1976; Sanchez-Palencia, 1980; Bakhvalov and Panasenko, 1984; Guedes and Kikuchi, 1990). This method has the capability of evaluating both macroscopic constitutive behavior and microscopic distributions of stress and strain, in contrast to simple methods effective only for the overall behavior of such composites. The method is therefore useful and has been applied to various problems starting with elastic one. For time-dependent deformation, however, only linear viscoelasticity and steady-state creep were dealt with previously (Sanchez-Palencia, 1980; Aravas et al., 1995; Shibuya, 1996; Yi et al., 1998).

The present authors then developed a homogenization theory for nonlinear time-dependent composites with periodic internal structures (Wu and Ohno, 1999; Ohno et al., 2000). This theory was formulated

* Corresponding author. Tel.: +81-52-789-4475; fax: +81-52-789-5131.
E-mail address: ohno@mech.nagoya-u.ac.jp (N. Ohno).

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without and with recourse to the asymptotic expansion in the macroscopically uniform and nonuniform cases, respectively. The theory is of the rate type so that the fields of displacement, stress, strain, etc. can be computed incrementally. In other words, on the assumption that the current fields of such variables are known, the fields of their rates are computed to know their fields after a small increment of time. This kind of rate-type formulation is in general used for nonlinear problems such as the homogenization of elastic–plastic and/or large deformation of periodic composites (e.g., Abeyaratne and Triantafyllidis, 1984; Agah-Tehrani, 1990; Terada et al., 1995; Okada et al., 1998).

The distributions of stress and strain may develop point symmetrically with respect to the center of each unit cell in periodic composites in many cases, as will be illustrated in Section 2. Then, a half of the unit cell can be sufficient for the incremental analysis based on the rate-type formulation. Nevertheless, because of the periodic condition to be satisfied on the unit cell boundary, the whole of a unit cell has been taken to be the domain of analysis even in recent studies including the present authors' works, in which the point symmetry of internal distributions with respect to the cell center is applicable (Terada et al., 1996; Okada et al., 1998; Wu and Ohno, 1999; Ohno et al., 2000). It is then worthwhile to discuss what replaces the periodic boundary condition in order to reduce the domain of analysis by half when the point symmetry prevails. Incidentally, appropriate boundary conditions were already shown for the line and hexagonal symmetries (Lene, 1984; Suquet, 1987).

This work deals with the homogenization of periodic composites in which the internal distributions develop point symmetrically with respect to the center of each unit cell. First, proving that perturbed velocity satisfies the point symmetry with respect to the cell boundary facet centers as well as the cell center, we point out that this symmetry can be utilized for prescribing the boundary condition of perturbed velocity and for reducing the domain of analysis to half a unit cell. Then, this finding is used to rebuild the homogenization theory of nonlinear time-dependent composites formulated by the present authors. Finally, as applications of the resulting theory, elastic–viscoplastic elongation of unidirectional composites is analyzed by assuming the square and hexagonal arrays of fibers subjected to either transverse or off-axial loading. It is thus shown that the point symmetry can be used for computing efficiently the inelastic behavior of nonlinear, periodic composites.

2. Property of perturbed velocity field

In this work, we consider a simple case in which an infinite body Ω with a periodic internal structure is subjected to macroscopically uniform stress and strain. Then, the homogenized behavior of Ω can be discussed by analyzing the smallest repeatable element, i.e., unit cell Y. Infinitesimal strain is assumed to take place in Ω macroscopically and microscopically.

Let y_i $(i = 1, 2, 3)$ be the local rectangular coordinates taken for Y and its vicinity. Then, the velocity field in Y and its vicinity at time t, $\dot{u}_i(\boldsymbol{y}, t)$, can have an expression

$$\dot{u}_i(\boldsymbol{y}, t) = \dot{F}_{ij}(t) y_j + \dot{c}_i(t) + \dot{u}_i^{\#}(\boldsymbol{y}, t), \tag{1}$$

where the superposed dot denotes the differentiation with respect to t, $F_{ij}(t)$ and $c_i(t)$ macroscopic deformation gradient and rigid displacement, respectively, and $u_i^{\#}$ the deviation of u_i from the macroscopic deformation including rigid displacement. From now on $u_i^{\#}$ is referred to as perturbed displacement. It is noted that $u_i^{\#}$ distributes periodically from unit cell to unit cell as a result of the periodic internal structure of Ω. This periodicity is called the Y-periodicity.

Using Eq. (1), strain rate is expressed as

$$\dot{e}_{ij} = \dot{E}_{ij} + e_{ij(y)}(\dot{\boldsymbol{u}}^{\#}), \tag{2}$$

where $\dot{E}_{ij}$ and $e_{ij(y)}(\dot{\boldsymbol{u}}^{\#})$ denote macroscopic and perturbed strain rates, respectively:

$$\dot{E}_{ij} = \tfrac{1}{2}(\dot{F}_{ij} + \dot{F}_{ji}), \tag{3}$$

$$e_{ij(y)}(\dot{\boldsymbol{u}}^{\#}) = \frac{1}{2}\left(\frac{\partial \dot{u}_i^{\#}}{\partial y_j} + \frac{\partial \dot{u}_j^{\#}}{\partial y_i}\right). \tag{4}$$

It is emphasized that perturbed strain rate is a consequence of the internal distributions of constituents, stress, strain and so on in Ω since no such internal distributions cause only $\dot{E}_{ij}$.

Let us now assume that the internal distributions, responsible for perturbed velocity, satisfy the symmetry with respect to the center of Y, C_0, in addition to the Y-periodicity, as illustrated in Fig. 1. Then, we can give a proof that the internal distributions also satisfy the symmetry with respect to the boundary facet centers of Y, C_i $(i = 1, 2, \ldots, N)$, where N indicates the total number of cell boundary facets. For the 2D unit cells, C_i $(i = 1, 2, \ldots, N)$ refer to the middle points of cell sides, and $N = 4$ and 6 when Y is square and hexagonal, respectively. The proof is simple. Let P be any point in Y, and let P_i and P_0 be the symmetric points of P with respect to C_i and C_0, respectively. Moreover, let C_i' be the symmetric point of C_i with respect to C_0 (Fig. 1). Then,

$$\overrightarrow{C_iP} + \overrightarrow{C_iP_i} = 0, \tag{5}$$

$$\overrightarrow{C_iP} + \overrightarrow{C_i'P_0} = 0. \tag{6}$$

Hence,

$$\overrightarrow{C_iP_i} = \overrightarrow{C_i'P_0}. \tag{7}$$

This equation means that P_0 and P_i take the positions at which the Y-periodicity is to be satisfied, and consequently that P_0 and P_i have the same internal states. Then, since P and P_0 have been assumed to have the same internal states, P and P_i also have the same. Therefore, the internal distributions satisfy the symmetry with respect to C_i $(i = 1, 2, \ldots, N)$ as well as C_0. Vice versa, if the internal distributions are symmetric with respect to C_0 and C_i $(i = 1, 2, \ldots, N)$, P_0 and P_i have the same internal states, so that the

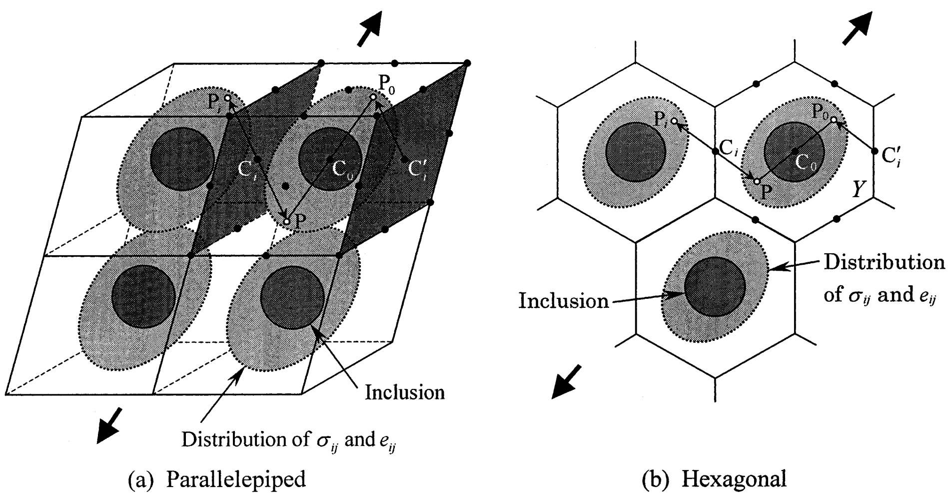

Fig. 1. Schematic illustration of internal distributions in periodic composites subjected to macroscopically uniform load: (a) parallelepiped and (b) hexagonal unit cells with points of symmetry for internal distributions.

internal distributions are Y-periodic. The point symmetry with respect to C_0 and C_i $(i = 1, 2, \ldots, N)$ will be referred to as the C-symmetry hereafter.

When the internal distributions satisfy the C-symmetry, the point-symmetric transformation of material points in Ω with respect to any of C_k $(k = 0, 1, \ldots, N)$ does not change the internal distributions. Moreover, the transformation does not affect macroscopic loading, as seen from Fig. 1. Therefore, the field of perturbed velocity $\dot{u}_i^{\#}$, which is induced as a consequence of the internal distributions in Ω under macroscopic loading, ought to satisfy the C-symmetry in addition to the Y-periodicity. Thus, we have

$$\dot{u}_i^{\#}(\boldsymbol{y}(\mathrm{P}_k), t) = -\dot{u}_i^{\#}(\boldsymbol{y}(\mathrm{P}), t), \quad k = 0, 1, \ldots, N, \tag{8}$$

where P_k indicates the symmetric point of P with respect to C_k. Especially, if P coincides with C_k, the above equation is reduced to

$$\dot{u}_i^{\#}(\boldsymbol{y}(\mathrm{C}_k), t) = 0, \quad k = 0, 1, \ldots, N. \tag{9}$$

Hence, $\dot{u}_i^{\#}$ vanishes at C_k $(k = 0, 1, \ldots, N)$ if the internal distributions have the C-symmetry.

Substitution of Eq. (8) into Eq. (2) gives

$$\dot{e}_{ij}(\boldsymbol{y}(\mathrm{P}_k), t) = \dot{e}_{ij}(\boldsymbol{y}(\mathrm{P}), t), \quad k = 0, 1, \ldots, N. \tag{10}$$

This in turn means the C-symmetry of stress rate $\dot{\sigma}_{ij}$ in Ω is

$$\dot{\sigma}_{ij}(\boldsymbol{y}(\mathrm{P}_k), t) = \dot{\sigma}_{ij}(\boldsymbol{y}(\mathrm{P}), t), \quad k = 0, 1, \ldots, N. \tag{11}$$

Incidentally, it can be shown that for parallelogram unit cells, the point symmetry of internal distributions and Eqs. (8)–(11) also hold at the cell vertices, and that for parallelepiped unit cells, they hold at the cell vertices and also at the middle points of cell edges, as seen from Fig. 1(a).

3. Introduction of semiunit cell

It has been shown in Section 2 that the field of perturbed velocity, $\dot{u}_i^{\#}(\boldsymbol{y}, t)$, satisfies the C-symmetry, i.e., Eq. (8), if the internal distributions responsible for $\dot{u}_i^{\#}$ have the symmetry with respect to C_0 in addition to the Y-periodicity. This suggests the introduction of a semiunit cell $\tilde{Y}$ (half a unit cell), on the boundary of which either the C-symmetry or the Y-periodicity can be imposed on $\dot{u}_i^{\#}$ as its boundary condition. Fig. 2(a) and (b) illustrates 2D examples of $\tilde{Y}$. In the example shown in Fig. 2(a), $\dot{u}_i^{\#}$ is prescribed to satisfy the C-symmetry on AB and CD and the Y-periodicity on AD and BC; in Fig. 2(b), the C-symmetry is imposed on $\dot{u}_i^{\#}$ on all the sides of $\tilde{Y}$.

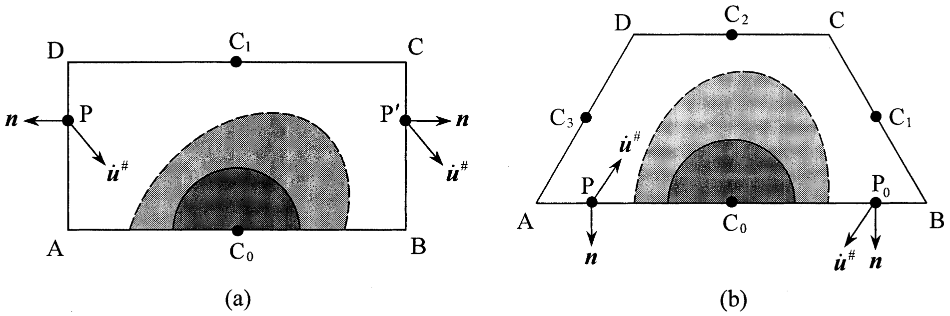

Fig. 2. Semiunit cell with boundary on which either C-symmetry or Y-periodicity of $\dot{u}_i^{\#}$ is prescribed: (a) C-symmetry on AB and CD, and Y-periodicity on AD and BC, and (b) C-symmetry on all sides.

Let $\tilde{\Gamma}$ be the boundary of $\tilde{Y}$. Then, we can show the following equation to be used in the subsequent section:

$$\int_{\tilde{\Gamma}} \dot{\sigma}_{ij}\tilde{n}_j\dot{u}_i^{\#}\,\mathrm{d}\tilde{\Gamma} = 0, \tag{12}$$

where $\tilde{n}_i$ denotes the unit vector outward normal to $\tilde{\Gamma}$. To see the above equation, let us divide $\tilde{\Gamma}$ into $\tilde{\Gamma}_{\mathrm{C}}$ and $\tilde{\Gamma}_Y$, on which the C-symmetry and the Y-periodicity of $\dot{u}_i^{\#}$ are prescribed as its boundary condition, respectively. For $\tilde{\Gamma}_{\mathrm{C}}$, each boundary facet has a pair of points P and P_k that are positioned symmetrically with respect to the facet center C_k (Fig. 2(b)). Then, using Eqs. (8) and (11), and noting that $\tilde{n}_i$ is the same at P and P_k, we can show that the boundary integral of $\dot{\sigma}_{ij}\tilde{n}_j\dot{u}_i^{\#}$ on $\tilde{\Gamma}_{\mathrm{C}}$ is zero. For $\tilde{\Gamma}_Y$, on the other hand, there exists a pair of points P and P′ at which the Y-periodicity is to be satisfied (Fig. 2(a)). Then, because of the Y-periodicity of $\dot{u}_i^{\#}$ and the Y-antiperiodicity of $\dot{\sigma}_{ij}\tilde{n}_j$, the boundary integral of $\dot{\sigma}_{ij}\tilde{n}_j\dot{u}_i^{\#}$ on $\tilde{\Gamma}_Y$ vanishes. We thus have Eq. (12). In the same way, we can show that

$$\int_{\tilde{\Gamma}} \dot{u}_i^{\#}\tilde{n}_j\,\mathrm{d}\tilde{\Gamma} = 0. \tag{13}$$

Incidentally, Shibuya (1996) noticed the symmetry of $\dot{u}_i^{\#}$ with respect to the middle points of cell sides in the special case of hexagonal unit cells. However, he did not show it, in general, on the basis of the symmetry with respect to cell centers and the Y-periodicity; moreover, he did not discuss about reducing the domain of analysis by introducing a semiunit cell, which is a matter of concern in the present work.

4. Homogenization theory for time-dependent deformation

In this section, employing the semiunit cell $\tilde{Y}$, introduced in Section 3, we rebuild the homogenization theory of time-dependent deformation developed by the present authors (Wu and Ohno, 1999; Ohno et al., 2000). Only the macroscopically uniform case, which does not need the asymptotic expansion of field variables, is considered for simplicity. We thus show that $\tilde{Y}$ allows the domain of analysis to be reduced by half without changing the resulting equations.

The equilibrium of $\dot{\sigma}_{ij}$ in Ω with no body force can be expressed as

$$\frac{\partial\dot{\sigma}_{ij}}{\partial y_j} = 0. \tag{14}$$

Let $v_i(\boldsymbol{y},t)$ be any variation of $\dot{u}_i^{\#}(\boldsymbol{y},t)$, so that $v_i(\boldsymbol{y},t)$ satisfies the C-symmetry and the Y-periodicity. Then, by integrating $(\partial\dot{\sigma}_{ij}/\partial y_j)v_i$ over $\tilde{Y}$, and by using the divergence theorem, the above equation is rewritten in an integral form

$$\int_{\tilde{Y}} \dot{\sigma}_{ij}e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y} - \int_{\tilde{\Gamma}} \dot{\sigma}_{ij}\tilde{n}_j v_i\,\mathrm{d}\tilde{\Gamma} = 0. \tag{15}$$

Since $v_i(\boldsymbol{y},t)$ is any variation of $\dot{u}_i^{\#}(\boldsymbol{y},t)$, Eq. (12) with $\dot{u}_i^{\#}(\boldsymbol{y},t)$ replaced by $v_i(\boldsymbol{y},t)$ is valid. Consequently, the second term in Eq. (15) vanishes. Thus, we have

$$\int_{\tilde{Y}} \dot{\sigma}_{ij}e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y} = 0. \tag{16}$$

Now, we assume that the constituents of Ω exhibit linear elasticity and nonlinear creep characterized by

$$\dot{\sigma}_{ij} = c_{ijkl}[\dot{\varepsilon}_{kl} - \beta_{kl}(\boldsymbol{\sigma})], \tag{17}$$

where c_{ijkl} and $\beta_{kl}(\boldsymbol{\sigma})$ indicate, respectively, the elastic constants and creep functions of constituents, and they satisfy

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \tag{18}$$

$$\beta_{kl} = \beta_{lk}. \tag{19}$$

Hereafter c_{ijkl} and $\beta_{kl}(\boldsymbol{\sigma})$, which can vary from constituent to constituent, will be regarded as functions of the coordinates y_i though not expressed explicitly.

Substitution of Eqs. (17) and (2) into Eq. (16) gives

$$\int_{\tilde{Y}} c_{ijpq} e_{pq(y)}(\dot{\boldsymbol{u}}^{\#}) e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y} = -\dot{E}_{kl} \int_{\tilde{Y}} c_{ijkl} e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y} + \int_{\tilde{Y}} c_{ijkl} \beta_{kl}(\boldsymbol{\sigma}) e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y}. \tag{20}$$

This equation poses the following boundary value problem to find $\dot{u}_i^{\#}(\boldsymbol{y}, t)$ in $\tilde{Y}$: Assuming that the current distribution of σ_{ij} in $\tilde{Y}$ is known, find the current field of perturbed velocity in $\tilde{Y}$, $\dot{u}_i^{\#}(\boldsymbol{y}, t)$, which satisfies Eq. (20) for any velocity field $v_i(\boldsymbol{y}, t)$ defined in $\tilde{Y}$; $\dot{u}_i^{\#}(\boldsymbol{y}, t)$ and its variation $v_i(\boldsymbol{y}, t)$ are subjected to either the C-symmetric or Y-periodic condition on $\tilde{\Gamma}$, as was illustrated in Fig. 2.

Now, let $\chi_i^{kl}(\boldsymbol{y}, t)$ and $\varphi_i(\boldsymbol{y}, t)$ be the functions which are determined by solving the following two boundary value problems for the semiunit cell $\tilde{Y}$, respectively:

$$\int_{\tilde{Y}} c_{ijpq} e_{pq(y)}(\boldsymbol{\chi}^{kl}) e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y} = \int_{\tilde{Y}} c_{ijkl} e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y}, \tag{21}$$

$$\int_{\tilde{Y}} c_{ijpq} e_{pq(y)}(\boldsymbol{\varphi}) e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y} = \int_{\tilde{Y}} c_{ijkl} \beta_{kl}(\boldsymbol{\sigma}) e_{ij(y)}(\boldsymbol{v})\,\mathrm{d}\tilde{Y}, \tag{22}$$

where $\chi_i^{kl}(\boldsymbol{y}, t)$ and $\varphi_i(\boldsymbol{y}, t)$ are prescribed to satisfy either the C-symmetric or Y-periodic condition on $\tilde{\Gamma}$, and $v_i(\boldsymbol{y}, t)$ is any velocity field satisfying this boundary condition. Then Eq. (20), in which $\dot{u}_i^{\#}(\boldsymbol{y}, t)$ depends on $\dot{E}_{kl}(t)$ linearly through the first term in the right-hand side, has a solution

$$\dot{u}_i^{\#}(\boldsymbol{y}, t) = -\chi_i^{kl}(\boldsymbol{y}, t)\dot{E}_{kl}(t) + \varphi_i(\boldsymbol{y}, t). \tag{23}$$

Substituting Eqs. (2) and (23) into Eq. (17), we obtain an evolution equation of σ_{ij}

$$\dot{\sigma}_{ij}(\boldsymbol{y}, t) = a_{ijkl}(\boldsymbol{y}, t)\dot{E}_{kl}(t) - r_{ij}(\boldsymbol{y}, t), \tag{24}$$

where

$$a_{ijkl} = c_{ijpq}[\delta_{pk}\delta_{ql} - e_{pq(y)}(\boldsymbol{\chi}^{kl})], \tag{25}$$

$$r_{ij} = c_{ijkl}[\beta_{kl}(\boldsymbol{\sigma}) - e_{kl(y)}(\boldsymbol{\varphi})]. \tag{26}$$

Here and from now on, δ_{ij} indicates Kronecker's delta. Let us introduce a volume average operator

$$\langle \# \rangle = \frac{1}{|\tilde{Y}|} \int_{\tilde{Y}} \#\,\mathrm{d}\tilde{Y}, \tag{27}$$

where $|\tilde{Y}|$ denotes the volume of $\tilde{Y}$. Eq. (24) then becomes a rate-type macroscopic constitutive equation

$$\dot{\Sigma}_{ij} = \langle a_{ijkl} \rangle \dot{E}_{kl} - \langle r_{ij} \rangle, \tag{28}$$

where

$$\langle \dot{\sigma}_{ij} \rangle = \dot{\Sigma}_{ij}. \tag{29}$$

Moreover, taking the volume average of Eq. (2) with Eq. (4), and using the divergence theorem and Eq. (13), we have

$$\langle \dot{e}_{ij} \rangle = \dot{E}_{ij}. \tag{30}$$

Eqs. (21)–(30), which enable us to compute the macroscopic constitutive behavior of Ω as well as the microscopic distributions of stress and strain in Ω, have been derived using the semiunit cell $\tilde{Y}$ instead of the unit cell Y. These equations, however, have the same forms, except for the boundary conditions of χ_i^{kl} and φ_i, as in the previous works (Wu and Ohno, 1999; Ohno et al., 2000), so that the incremental computational procedure described there is also useful in the present work: Let us suppose that the history of either Σ_{ij} or E_{ij}, or a combination of them, is prescribed, and that the distributions of σ_{ij} and e_{ij} in $\tilde{Y}$ are known at the current time t. Eqs. (21) and (22), can then be solved to find $\chi_i^{kl}(\boldsymbol{y}, t)$ and $\varphi_i(\boldsymbol{y}, t)$ using a finite element method. When $\chi_i^{kl}(\boldsymbol{y}, t)$ and $\varphi_i(\boldsymbol{y}, t)$ are found, Eq. (28) with the prescribed components of $\dot{\Sigma}_{ij}$ and $\dot{E}_{ij}$ allows us to determine the unknown components of $\dot{\Sigma}_{ij}$ and $\dot{E}_{ij}$. Subsequently Eqs. (24) and (17) are used to determine $\dot{\sigma}_{ij}(\boldsymbol{y}, t)$ and $\dot{e}_{ij}(\boldsymbol{y}, t)$. Then, calculating the increments from t to $t + \Delta t$ as $\Delta\Sigma_{ij} = \dot{\Sigma}_{ij}\Delta t$, $\Delta\sigma_{ij} = \dot{\sigma}_{ij}\Delta t$, etc., and adding the increments to the current values, we proceed to the succeeding time step.

Incidentally, Eqs. (21)–(30) can be cast into incremental forms appropriate for finite values of Δt in the same way as in the previous works (Wu and Ohno, 1999; Ohno et al., 2000).

5. Examples of numerical analysis

In order to verify the present theory based on semiunit cells, we apply the theory to analyze the elastic–viscoplastic properties of unidirectional fiber-reinforced composites subjected to transverse and off-axial loading. Unit cells were used to analyze the properties previously (Ohno et al., 2000).

5.1. Material system

We consider two types of fiber arrangements, square and hexagonal arrays, shown in Fig. 3(a) and (b), respectively. The y_1- and y_2-axes are taken in a plane perpendicular to the fiber orientation, as shown in the figures, and the y_3-axis is directed upward from the plane. The fibers, which are infinitely long and occupy the volume fraction of V_f in the composites, are assumed to obey Hooke's law

$$e_{ij} = \frac{1 + \nu_f}{E_f}\sigma_{ij} - \frac{\nu_f}{E_f}\sigma_{kk}\delta_{ij}, \tag{31}$$

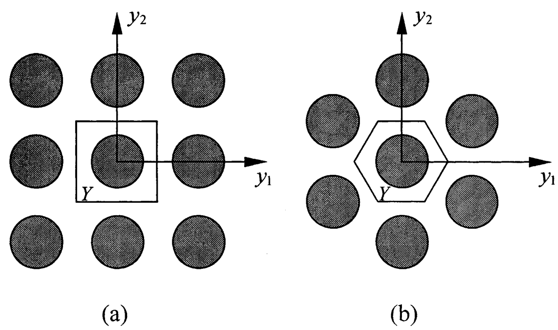

Fig. 3. Fiber array and coordinates: (a) square array and (b) hexagonal array.

where E_f and ν_f are material constants. The matrix, on the other hand, is assumed to exhibit power-law creep in addition to linear elasticity, i.e.,

$$\dot{e}_{ij} = \frac{1+\nu_m}{E_m}\dot{\sigma}_{ij} - \frac{\nu_m}{E_m}\dot{\sigma}_{kk}\delta_{ij} + \frac{3}{2}A\sigma_{\rm eq}^{n-1}s_{ij}, \tag{32}$$

where E_m, ν_m, A and n are material constants, s_{ij} indicates the deviatoric part of σ_{ij}, and $\sigma_{\rm eq} = [(3/2)s_{ij}s_{ij}]^{1/2}$.

Let us introduce the following nondimensional stresses, strains and time for convenience in computation:

$$\sigma^*_{ij} = \frac{\sigma_{ij}}{\sigma_0}, \quad \Sigma^*_{ij} = \frac{\Sigma_{ij}}{\sigma_0}, \quad e^*_{ij} = \frac{E_f e_{ij}}{\sigma_0}, \quad E^*_{ij} = \frac{E_f E_{ij}}{\sigma_0}, \quad t^* = AE_f\sigma_0^{n-1}t, \tag{33}$$

where σ_0 indicates an appropriate reference stress. Eqs. (31) and (32) are then nondimensionalized as

$$e^*_{ij} = (1+\nu_f)\sigma^*_{ij} - \nu_f\sigma^*_{kk}\delta_{ij}, \tag{34}$$

$$\frac{\partial e^*_{ij}}{\partial t^*} = \frac{E_f}{E_m}\left[(1+\nu_m)\frac{\partial \sigma^*_{ij}}{\partial t^*} - \nu_m\frac{\partial \sigma^*_{kk}}{\partial t^*}\delta_{ij}\right] + \frac{3}{2}(\sigma^*_{\rm eq})^{n-1}s^*_{ij}, \tag{35}$$

where s^*_{ij} indicates the deviatoric part of σ^*_{ij}, and $\sigma^*_{\rm eq} = [(3/2)s^*_{ij}s^*_{ij}]^{1/2}$. As a consequence, the material parameters to be specified in computation are E_f/E_m, ν_f, ν_m and n in addition to V_f. Assumed for them are the following constants concerned with a continuous SiC fiber/titanium alloy composite (Kroupa and Neu, 1994):

$$\frac{E_f}{E_m} = 4, \quad \nu_f = 0.25, \quad \nu_m = 0.34, \quad n = 4, \quad V_f = 0.395. \tag{36}$$

5.2. Finite element method and meshes

Fig. 4(a) and (b) illustrates the unit cells employed in this work for the square and hexagonal arrays of fibers, respectively. As shown in the figures, their upper halves are taken as the semiunit cells and divided into finite elements, which are isoparametric elements with four nodes. The small solid circles in the figures indicate the nodal points at which χ_i^{kl} and φ_i are set to be zero due to the C-symmetric boundary condition, as was shown by Eq. (9) and illustrated in Fig. 2(a) and (b).

In the finite element programming to solve Eqs. (21) and (22), it is necessary to deal with the boundary condition of χ_i^{kl} and φ_i, i.e., either the C-symmetry or the Y-periodicity of χ_i^{kl} and φ_i on $\tilde{\Gamma}$. The C-symmetry

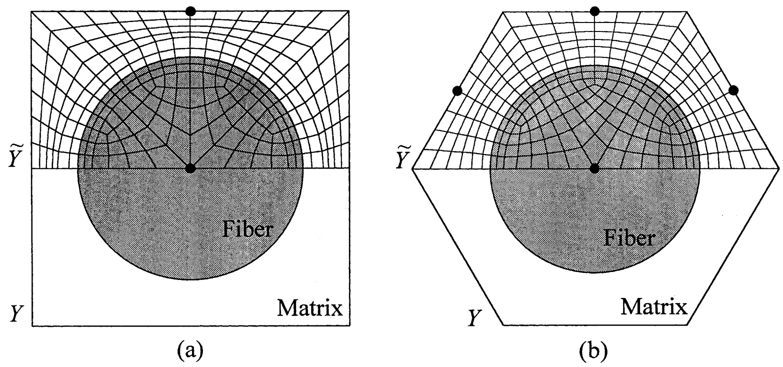

Fig. 4. Unit cell Y and finite element mesh of semiunit cell $\tilde{Y}$: (a) square array and (b) hexagonal array.

of χ_i^{kl} and φ_i on $\tilde{\Gamma}$, which is new, however, can be programmed in the same way as the Y-periodicity using the penalty method. Hence the program of finite element analysis in the previous work (Ohno et al., 2000), in which some details of the program were described especially with respect to the Y-periodicity, has been extended further to incorporate the C-symmetric boundary condition.

The program mentioned above allows us to analyze semi-3D deformation by means of 2D finite element meshes. Each node has three nodal values to represent the y_1-, y_2- and y_3-components of χ_i^{kl} and φ_i. We, however, assume that deformation is semi-3D, i.e., uniform in the fiber direction even microscopically. Then, since $\partial\chi_i^{kl}/\partial y_3 = 0$ and $\partial\varphi_i/\partial y_3 = 0$ $(i = 1, 2, 3)$, 3D finite element meshes are not necessary (Noguchi and Shimizu, 1999; Ohno et al., 2000).

5.3. Loading condition

Let the composites be subjected to the following two types of macroscopic loading to examine the present theory.

(1) *Transverse loading*: This is the macroscopic loading in the y_1–y_2 plane:

$$\dot{E}^*_{\xi\xi} = 1, \qquad \Sigma^*_{\xi\eta} = \Sigma^*_{\eta\eta} = 0, \qquad E^*_{33} = E^*_{3\xi} = E^*_{3\eta} = 0, \tag{37}$$

where the plane-strain condition in the fiber direction is assumed, and ξ and η denote the loading coordinate axes making an angle θ with the y_1- and y_2-axes (Fig. 5(a)).

(2) *Off-axial loading*: The composites are subjected to the uniaxial loading given in a direction deviating from the y_3-axis by an angle φ:

$$\dot{E}^*_{\zeta\zeta} = 1, \qquad \Sigma^*_{\xi\xi} = \Sigma^*_{\xi\zeta} = 0, \qquad \Sigma^*_{22} = \Sigma^*_{2\xi} = \Sigma^*_{2\zeta} = 0, \tag{38}$$

where ξ and ζ denote the loading coordinate axes obtained by rotating the y_1- and y_3-axes by φ with respect to the y_2-axis (Fig. 5(b)).

5.4. Results of analysis

We, now show the results of analysis using the nondimensional macroscopic equivalent stress and strain defined as

$$\Sigma^*_{\mathrm{eq}} = \left(\tfrac{3}{2}S^*_{ij}S^*_{ij}\right)^{1/2}, \qquad E^*_{\mathrm{eq}} = \left(\tfrac{2}{3}E^*_{ij}E^*_{ij}\right)^{1/2}, \tag{39}$$

where S^*_{ij} indicates the deviatoric part of nondimensional macroscopic stress Σ^*_{ij}.

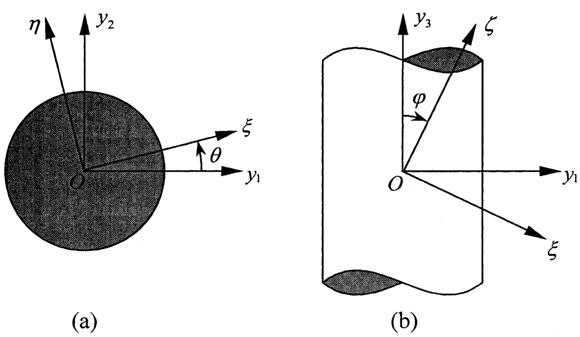

Fig. 5. Loading direction: (a) transverse loading and (b) off-axial loading.

Let us start with the case of transverse loading. Fig. 6(a) and (b) shows the Σ^*_{eq} versus E^*_{eq} relations obtained for the square and hexagonal fiber arrays under this loading, respectively. The same results were obtained by taking the unit cells shown in Fig. 4(a) and (b) as the domains of analysis (Ohno et al., 2000). Fig. 7(a) and (b) depicts the deformed meshes of the semiunit cells at $E^*_{eq} = 20$ under the transverse loading of $\theta = 45°$. They are just the upper halves of the deformed meshes of unit cells computed in the previous work (Ohno et al., 2000). Therefore, we can say that the formulation based on semiunit cells is successful, and that the C-symmetric boundary condition works well.

It is seen from Fig. 6(a) and (b) that, under transverse loading, the square and hexagonal fiber arrays exhibit significant and little anisotropy in the macroscopic behavior, respectively. This feature was discussed in detail in the case of creep by Wu and Ohno (1999).

Now, we proceed to the case of off-axial loading. For this loading, we discuss only the hexagonal array of fibers to save the space. Fig. 8 shows the Σ^*_{eq} versus E^*_{eq} relations obtained for the hexagonal array. This result is completely the same as the previous one (Ohno et al., 2000), which was computed using the unit cell Y shown in Fig. 4(b). Fig. 9(a) illustrates the deformed mesh of the semiunit cell $\tilde{Y}$ at $E^*_{eq} = 20$ under the off-axial loading of $\varphi = 20°$. This deformed mesh also coincides with the previous result (Ohno et al., 2000), which is shown in Fig. 9(b) for reference. We thus have confirmed again the validity of the present theory rebuilt by employing the semiunit cell $\tilde{Y}$ and the C-symmetric boundary condition.

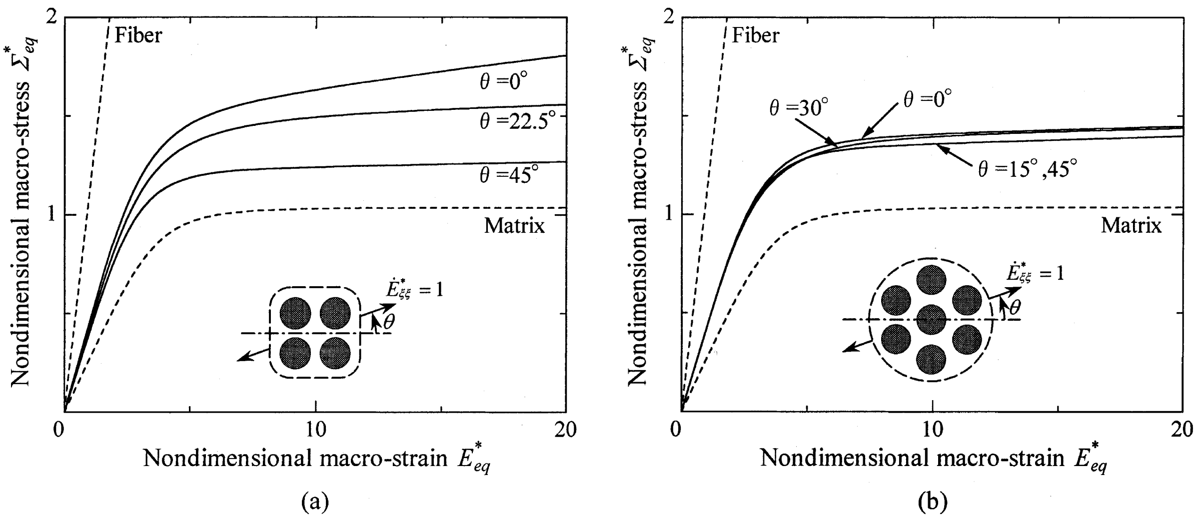

Fig. 6. Relation of nondimensional equivalent macro-stress and macro-strain under transverse loading: (a) square array and (b) hexagonal array.

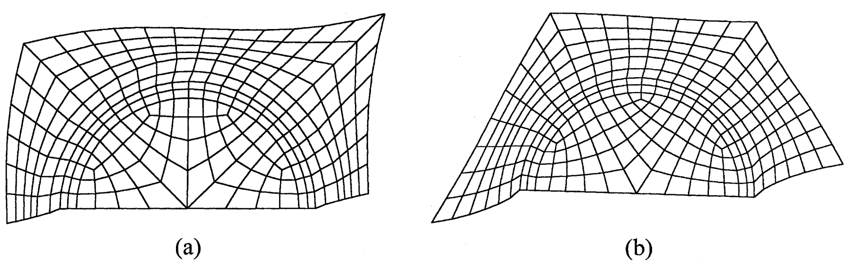

Fig. 7. Deformation of semiunit cell at $E^*_{eq} = 20$ under transverse loading of $\theta = 45°$: (a) square array and (b) hexagonal array.

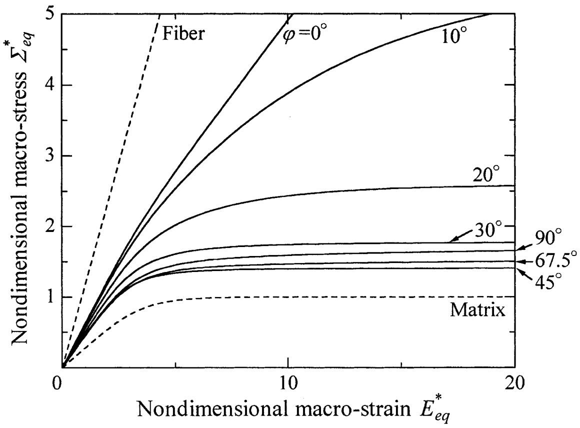

Fig. 8. Relation of nondimensional equivalent macro-stress and macro-strain of hexagonal array under off-axial loading.

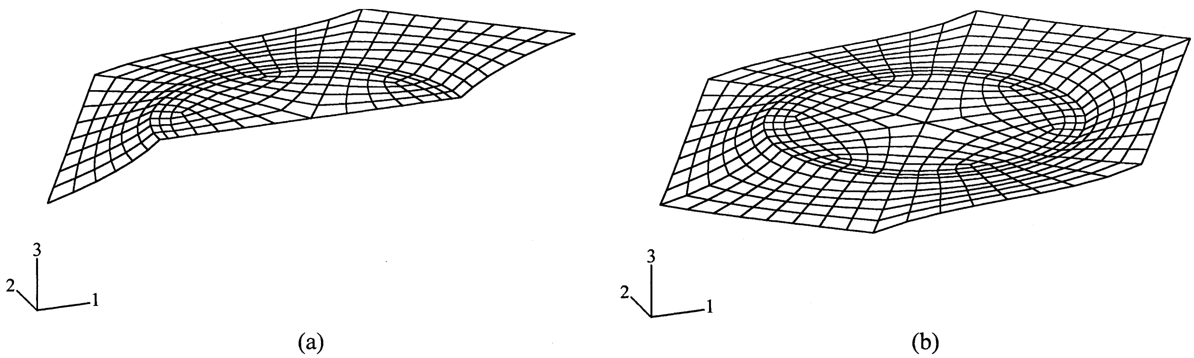

Fig. 9. Deformation of cell at $E^*_{eq} = 20$ under off-axial loading of $\varphi = 20°$: (a) semiunit cell and (b) unit cell.

It is seen from Fig. 8 that such a small value as $\varphi = 10°$ gives noticeable nonlinearity to the macroscopic stress–strain relation, and that the increase of φ beyond 30° does not significantly affect the macroscopic tensile behavior. Such dependence on φ was observed for e.g. in the tensile experiments of a hybrid composite GLARE 2 (Kawai et al., 1998).

6. Conclusions

In this work, on the assumption that the internal distributions of constituents, stress, strain, etc. affecting perturbed velocity are symmetric with respect to the unit cell centers in periodic composites, the symmetry of perturbed velocity fields, the domain of analysis, and the boundary condition were discussed. Then, the homogenization theory developed for nonlinear time-dependent composites by the present authors (Wu and Ohno, 1999; Ohno et al., 2000) was rebuilt and applied to analyze the elastic–viscoplastic properties of unidirectional composites. The following results were thus obtained.

First, it was shown that if the internal distributions have the symmetry with respect to the unit cell centers in periodic composites, they also have the symmetry with respect to the cell boundary facet centers. Then, by taking notice that perturbed velocity satisfies the symmetry with respect to any of the two kinds of centers, semiunit cells were introduced so that either this point symmetry, called C-symmetry, or the Y-periodicity can be prescribed as the boundary condition of perturbed velocity. Using this idea, the homogenization theory of nonlinear time-dependent composites was reformulated in order to reduce the domain of analysis by half. The resulting theory, which turned out to have no formal difference from the previous one except for the boundary condition, was verified by analyzing elastic–viscoplastic, unidirectional composites subjected to transverse and off-axial loading.

Finally, let us point out that computation time can be reduced at most to about one eighth by adopting semiunit cells, since such cells allow the number of finite elements to be decreased by half.

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